B.Sc. Part III (H). 7th Paper Diff. Eqns.

SODEVC (contd.)

Q. Solve

$$(x\sin x+\cos x)\frac{d^2y}{dx^2}-x\cos x\frac{dy}{dx}+y\cos x=0$$

Soln. The given eqn

$$(x\sin x+\cos x)\frac{d^2y}{dx^2}-x\cos x\frac{dy}{dx}+y\cos x=0$$

$$\Rightarrow \frac{d^2y}{dx^2}-\frac{x\cos x}{x\sin x+\cos x}\frac{dy}{dx}+\frac{\cos x}{x\sin x+\cos x}y=0 \quad \text{---(1)}$$

It is of the form $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$

Here, $P=\frac{-x\cos x}{x\sin x+\cos x}$, $Q=\frac{\cos x}{x\sin x+\cos x}$, $R=0$. ---(2)

Now, $P+Qx=\frac{-x\cos x}{x\sin x+\cos x}+\frac{x\cos x}{x\sin x+\cos x}=0$

$\Rightarrow u=x$ ---(3) is a part of the CF of soln. of (1).

Let $y=uv$ be the complete solution.

Then, v is given by

$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u}$$

Now, using the values from (2), the above eqn becomes

$$\frac{d^2v}{dx^2}+\left(\frac{-x\cos x}{x\sin x+\cos x}+\frac{2}{x}\cdot 1\right)\frac{dv}{dx}=0 \quad \text{---(4)}$$

$\left(\because u=x\Rightarrow \frac{du}{dx}=1.\right)$

Put $\frac{dv}{dx} = z \Rightarrow \frac{d^2v}{dx^2} = \frac{dz}{dx}$.

$$\Rightarrow \frac{dz}{dx} + \left(\frac{-x\cos x}{x\sin x + \cos x} + \frac{2}{x} \right) z = 0$$

$$\Rightarrow \frac{dz}{z} + \left(\frac{-x\cos x}{x\sin x + \cos x} + \frac{2}{x} \right) dx = 0$$

Integrating, we get

$$\Rightarrow \int \frac{dz}{z} = \int \frac{x\cos x}{x\sin x + \cos x}\, dx - 2\int \frac{1}{x}\, dx$$

$$\Rightarrow \log z = \log(x\sin x + \cos x) - 2\log x + \log c_1$$

$$\Rightarrow z = \frac{x\sin x + \cos x}{x^2}\, c_1$$

But $z = \frac{dv}{dx}$

$$\therefore \frac{dv}{dx} = \frac{x\sin x + \cos x}{x^2}\, c_1 = \left(\frac{\sin x}{x} + \frac{\cos x}{x^2} \right) c_1$$

$$\Rightarrow dv = c_1 \frac{\sin x}{x}\, dx + c_1 \frac{\cos x}{x^2}\, dx$$

Integrating, we get

$$\Rightarrow v = c_1 \int \frac{\sin x}{x}\, dx + c_1 \int \frac{\cos x}{x^2}\, dx$$

$$\Rightarrow v = c_1 \left[\frac{1}{x} \int \sin x\, dx - \int \left[\frac{d}{dx}\left(\frac{1}{x}\right) \int \sin x\, dx \right] dx \right] + c_1 \int \frac{\cos x}{x^2}\, dx$$

$$= c_1 \left[-\frac{1}{x}\cos x - \cancel{\int \frac{1}{x^2} \cos x\, dx} \right] + \cancel{c_1 \int \frac{\cos x}{x^2}\, dx} + c_2$$

$\Rightarrow v = -\frac{c_1}{x}\cos x + c_2$. Hence $y = uv$ where $u = x$, $v = -\frac{c_1}{x}\cos x + c_2$ is the complete soln.